Sequencing in Elementary Mathematics

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Abstract

These are the (combined) notes for two presentations given at the Core Knowledge National Conference in Atlanta on March 5, 2004. More detailed versions of the topics discussed here, and much more, can be found in the textbook

Elementary Mathematics for Teachers

written with with S. Baldridge and available at www.SingaporeMath.com

1. Place value and Column Addition
2. Sample operation: Addition
3. Bar Diagrams
4. Fractions

These notes include boxes containing

Teaching Comments

Some are very specific. Others state the essential mathematical principle at play; getting students to understand those principles should be a "teaching goal" for a lesson, series of lessons, or a particular topic.

Place Value

The idea of numbers is natural, but our way of writing them *is not natural.* To see that, consider three historical systems:

Tallies. (used by cavemen)

$$|,\ ||,\ |||,\ ||||,\ \ldots$$

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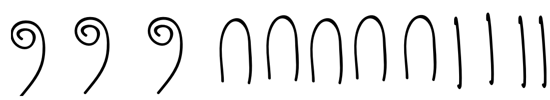
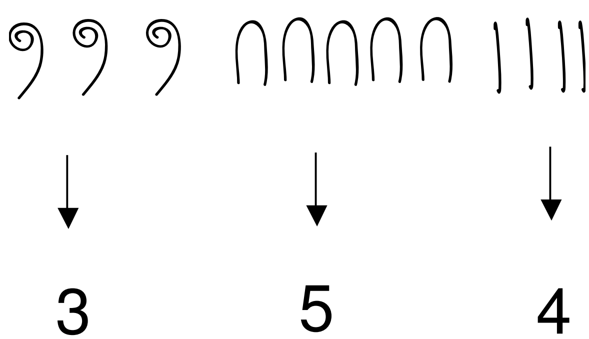
Egyptian Numerals.

Tallies to 9, then $\cap$ for 10 tallies, and  for 100.



Decimal Numerals. Ten digits suffice! The trick:



This idea of using position to encode value is a *secret code*. Because this is second nature to adults, it is easy to overlook the fact that it is *difficult to learn*. It requires explicit teaching and must be repeatedly worked on. Place value ideas occur and present difficulties in topic after topic in elementary school.

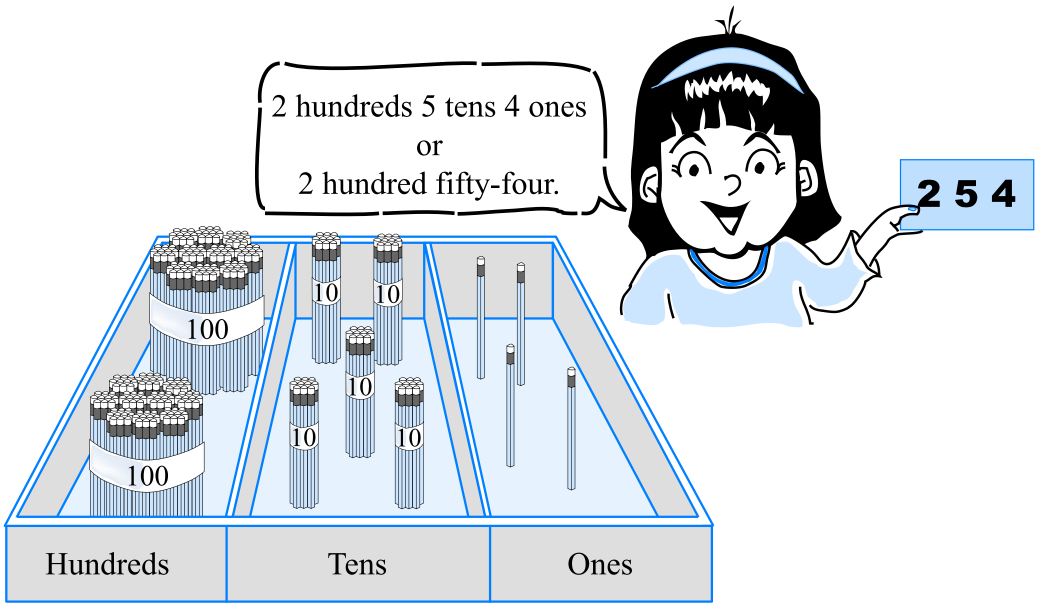
Place value causes MANY, MANY problems all through elementary school

A simple Grade K-1 Place Value exercise

Open "Word", set the font size large (24 point). Have the child start typing, preferably at the numeric end of the keyboard,

0 _ _ 1 _ _ 2 _ _ 3 _ _ ...

with a parent or teacher saying the number aloud as the child types it, correcting errors, and giving hints. The child should keep going, a few minutes a day, day after day. Any child who reaches 1000 will have a good understanding of the decimal numeral system.



The Place Value Process

(i) Form bundles of 1, 10, 100, 1000 etc.

(ii) *Rebundle* whenever there are more than 9 some denomination.

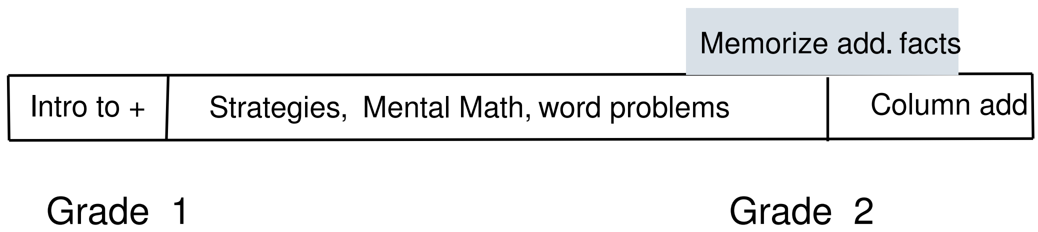
(iii) Count the bundles of each denomination and record that number in the appropriate position.

Steps (i) and (iii) — EASY Steps (ii) required — HARD.

Mental Math:

- What is 10 more than 74? 100 more than 281?
- What is 30 more than 241? 500 less than 784?
- What is 3 more than 59? 20 less than 86?
  These hard — require step (ii)

Addition Facts Grades 1-2



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Strategies

Not 100 facts to memorize! Instead, *learn how addition works* and how it interacts with PV.

Adding +1, +2	$6 + 2 = 8$	easy by counting-on
Adding +0	$5 + 0 = 5$	trivial, once taught
Commutativity	$2 + 7 = 7 + 2$	pointed out repeatedly.

Commutativity is not obvious!

Doubles	$3+3,\ 4+4,\ \ldots,\ 9+9$	learned by practice

Give facts "personalities" : $5+5=10$ 'fingers fact'
$6+6=12$ "egg-carton trick".

Tens Combinations	$5+5,\ 6+4,\ 7+3,$ $8+2,\ 9+1$	learned with practice
Adding 10	$7+10=17$	pointed out and used

These PV skills (steps toward multi-digit addition) are not obvious! *Spend time teaching these.*

Last come the "2-step" strategies:

Relate to Doubles:	$6+7=(6+6)+1=13$	MM practice
or to 10:	$9+6=(9+1)+5=15$	MM practice

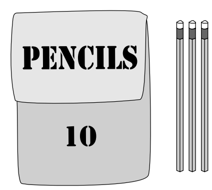
Altogether, only four facts not covered $(5+3, 6+3, 7+4, 7+5)$!

Get parents involved!

Word Problems

Simple word problems can be done in the early grades. Very few words are required!

7 − 3 = __



Ryan has 13 pencils.
He buys 4 more.
How many pencils has he now?

□ ○ □ = □

He has ___ pencils now.

Do not hold progress in math hostage to reading or writing ability. There is much math that can be done in the early grades with a minimal use of reading and writing. Keep problems short and focused on math!

Column Add. Teaching Sequence

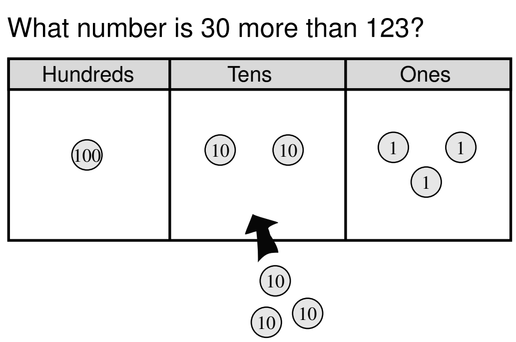
For each of the four operations learning the algorithm is the culmination of topic — after that the the topic is *done*.

The mathematics determines a "teaching sequence"

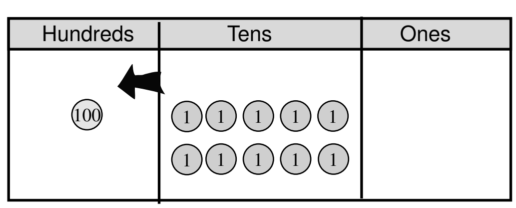
Prerequisites:

- Count to 1000
- Many 1-digit addition facts.
- 2-digit MM, including 20+30, 200+30, 200+300
- PV ideas of

(a)



(b) 10 dimes = 1 dollar



Step 1 Addition without rebundling. Simple: separately add hundreds, tens, ones

$$\begin{array}{r} 231 \\ + \quad 524 \\ \hline \end{array}$$

Hundreds	Tens	Ones

Step2 Addition without rebundling. Simple: separately add hundreds, tens, ones

$$\begin{array}{r} 27 \\ + \quad 46 \\ \hline \end{array}$$

Hundreds	Tens	Ones

Then increase complication, visiting all cases:

(b) 10 tens = 1 hundred

(c) 10 hundreds = 1 thousand

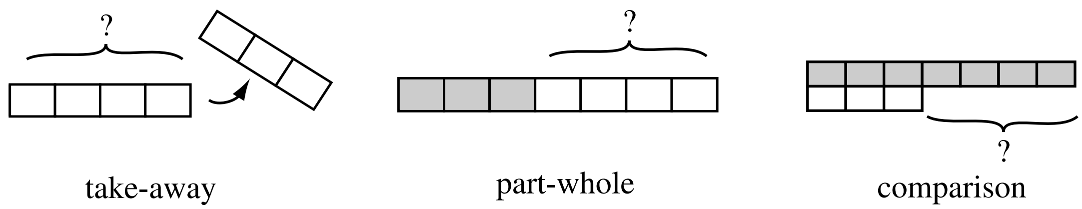
(d) Both (b) and (c)

(e) Students continue on their own (with checking!)

Bar Diagrams

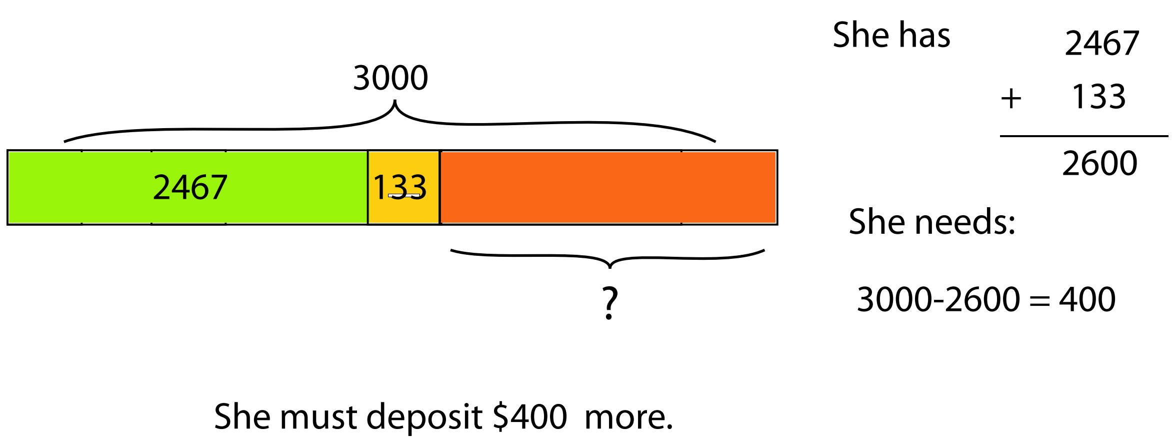
Bar diagrams help students recognize *when to use* arithmetic operations.

Three uses of subtraction for 7-3:



Students learn to (i) link each diagram with subtraction
(ii) construct diagrams for word problems.

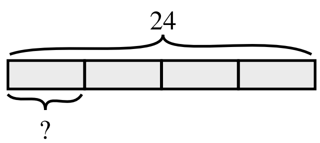
First one-step, then multistep problems: Susan had $2467 in a bank. She deposited another $133. How much more money must she deposit if she wants to have $3000?



The question gives 3 numbers — what do we do with them? The diagram summaries the question completely and shows what to do. Notice that bar diagrams need not be to scale.

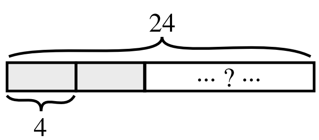
Similarly, two uses of division:

(a) There were 24 desks to clean. 4 boys shared the work equally. How many desks did each boy clean?



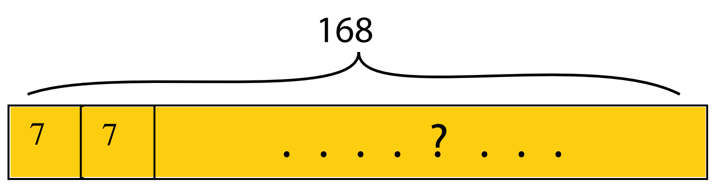
24 is 4 groups of size ___.

(b) 24 candies were packed in boxes of 4. How many boxes were there?



24 is ___ groups of size 4 .

Example Mr Chen wants to buy umbrellas. Each umbrella costs \$7. How many can he buy with \$168?



The diagram shows that the question is "How many 7's make 168?"; students should recognize that as asking for $168 \div 7$, which they can find by long division.

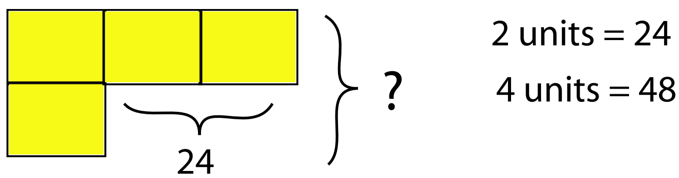
The bar diagram should *not* be used to answer the question by drawing in lots and lots of boxes of size 7. The *only* purpose of the diagrams is to help students recognize what arithmetic must be done. The arithmetic itself is then done without the diagram.

In particular, when a student sees what to do, there is no need for a diagram.

Bar diagrams become more powerful, and illustrate new concepts, as the grades progress.

(Grade 3) There are 3 times as many boys as girls. If there are 24 more boys than girls, how many children are there altogether?

A labeled bar diagram, with a ? completely summarize the problem"



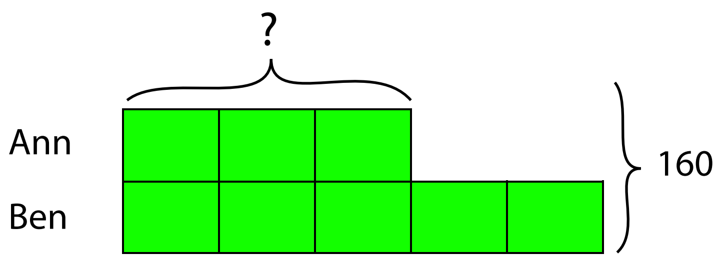
There were 48 children altogether.

(Grade 5) Peter has twice as much money as Joe. Joe has \$40 than Emily. They have \$300 altogether. How much money does Peter have?

Try this on your own!

Bar diagrams work wonderfully for ratios!

(Grade 5) The ratio of Ann's weight to Ben's weight is 3:5. Together they weigh 160 lbs. . How much does Ann weigh?

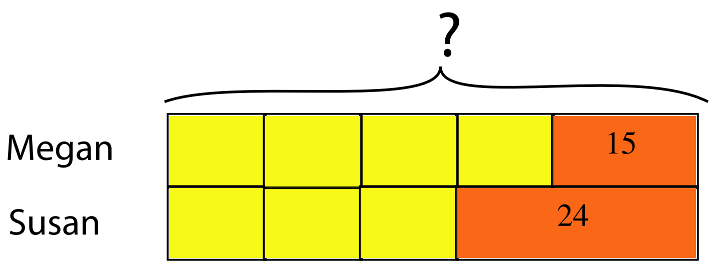


8 units = 160
1 unit = 20
3 units = 60.

Ann weighs 60 lbs.

(Grade 6) Megan and Susan's had equal amount of money. After Megan spent \$15 and Susan spent \$24, the ratio of Megan's money to Susan's money was 4:3. How much money did each girl have at first?

Sometimes with before-and-after situations it is best to start by constructing the diagram for the *after* situation (yellow), then add to it to get the before solution.



1 unit = 24 -15 = 9

Megan had:

4 units + 15 = 36+15

= 51.

Each girl had \$51 at first.

Fractions

When we count or measure we always have some *unit* in mind.

Emphasize Units!

3 feet, 80 lbs, 4 kegs

We use fractions when there is a unit (the *whole unit*) but we want to measure in terms of a smaller unit (the *fractional unit*).

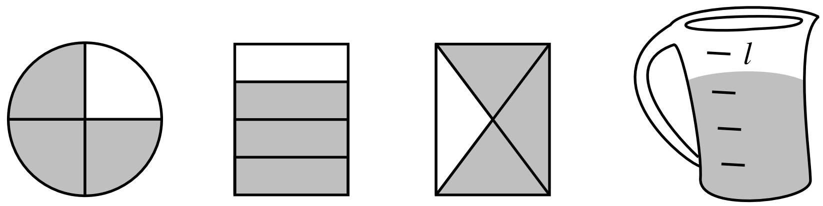
• **Grades 1 & 2** Use fractions *verbally* (no written fractions):

1 fifth + 2 fifths = 3 fifths.

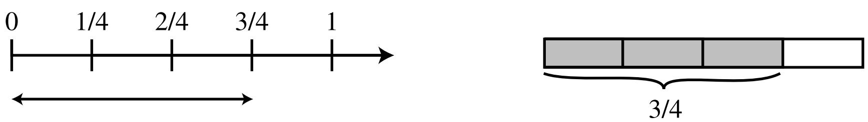
Once we fix the fractional unit, we count, add and subtract just as before

- **Pictures**

Area or Regional Model

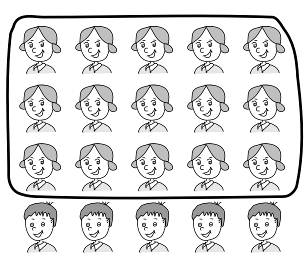


Linear Measurement Model



Exercise: read rulers, clocks, scales with *unlabeled* fractional rulings.

Set Model — **Not good for fractions**



$\frac{3}{4}$ of the students are girls.

- **Notation** for fractions is unnatural and confusing.

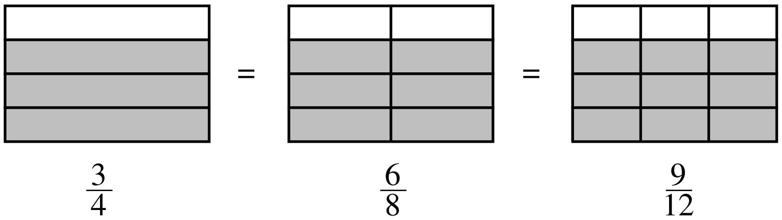
$$\frac{1}{5} + \frac{2}{5} \;=$$

3 quarters $\quad \begin{matrix} 3 & \leftarrow \text{number of fractional units} \\ \hline 4 & \leftarrow \text{name of fractional unit (number in whole unit)} \end{matrix}$

Its the notation, not the fractions, that is difficult

Primary Math 2B pg. 55–57

- **Equivalent Fractions**: Use rectangles, not pies!



$$\frac{2}{3} = \frac{}{6} = \frac{8}{}$$

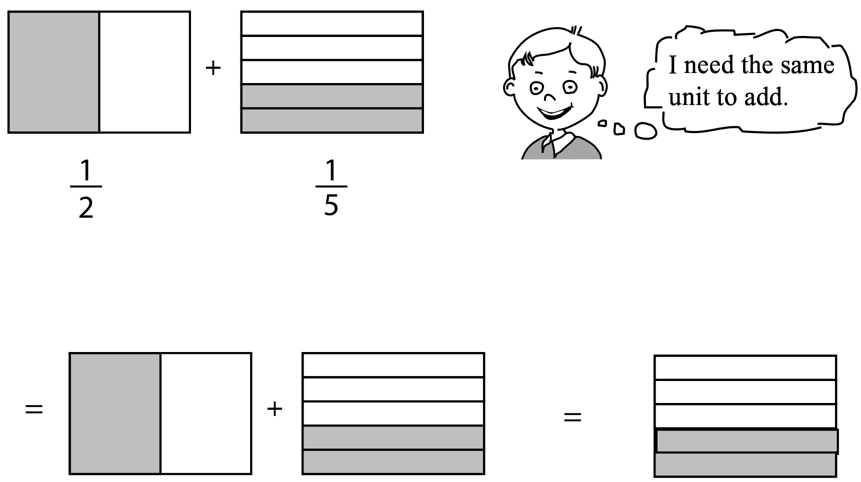
The teaching goal is to do it without diagrams.

Addition & subtraction: We can't add with different units:

$$2 \text{ cups} + 4 \text{ gallons} \ = ?$$

must
— Express in common unit
— Then count, add, and subtract as before.

(1)



(2) $\dfrac{1}{4} + \dfrac{2}{3} = \dfrac{3}{12} + \dfrac{8}{12} = \dfrac{11}{12}$

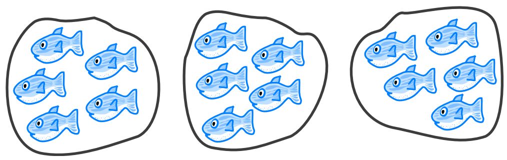
Move to numbers only asap

(3) Preparation for

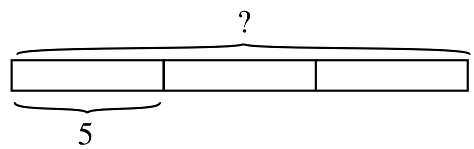
$$\frac{3}{x} + \frac{5}{y} = \frac{3y}{6xy} + \frac{5x}{6xy} = \frac{3y + 5x}{6xy}$$

Multiplication & division: For whole numbers, 3×5 means

3 groups of 5

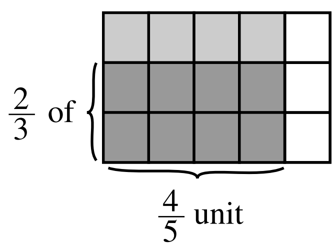


3 units of size 5



For multiplications, only the measurement model carries over to fractions. Then best illustrated by chopping vertically and horizontally to create rectangular arrays:

$$\frac{2}{3} \times \frac{4}{5} = \frac{2}{3} \text{ of } \frac{4}{5} = \frac{8}{15}$$



One then sees shortcut:

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{\text{(doubly shaded squares)}}{\text{(total number of squares)}}$$

Grade 5A A vendor sold $\frac{2}{3}$ of his hot dogs in the morning and $\frac{1}{6}$ in the afternoon. He sold 200 altogether. How many hot dogs did he have left?

Try this with a bar diagram.

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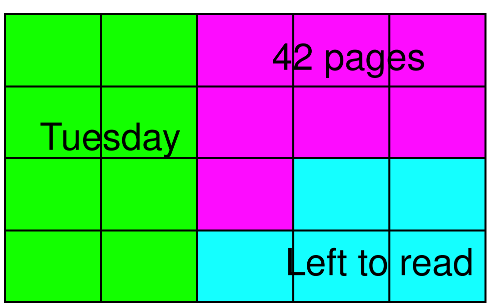
Linear pictures work when one denominator is a multiple of the other (subdivide)

Grade 6A Sara read 42 pages on Monday. She read $\frac{2}{5}$ of the book on Tuesday. If she still had $\frac{1}{4}$ of the book left to read, how many pages were there in the book?

Both fifths and quarters are used. Divide a rectangle vertically in fifths and horizontally into fourths. Shade the $\frac{2}{5}$ she read on Tuesday, and the $\frac{1}{4} = \frac{5}{20}$ left to read. The remaining 7 (purple) units correspond to the 42 pages.



7 fractional units = 42

1 fractional unit = 6

20 fractional units = 120

The book had 120 pages.

Rectangular arrays are useful for fraction addition also.